Adaptive second order central schemes on unstructured staggered grids

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Abstract

This communication is devoted to the derivation and numerical implementation of higher order central schemes on adaptive unstructured grids for approximating nonlinear hyperbolic conservation laws in several spatial dimensions.

1 Introduction

We consider the following Cauchy problem for the unknown function $u : \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$, $d \geq 1$:

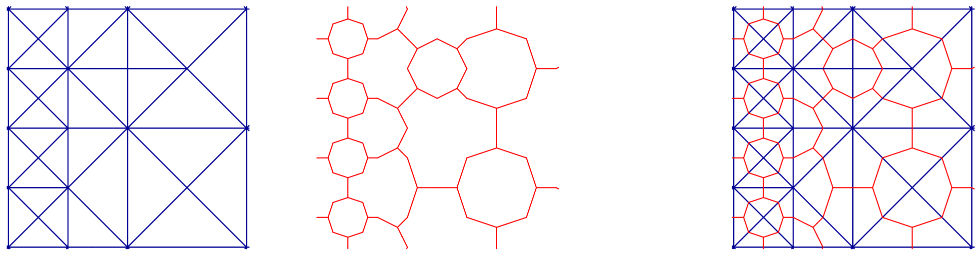
$$\partial_t u(x,t) + \operatorname{div} \mathrm{f(u(x,t))} = 0 \quad (x,t) \in \mathbb{R}^d \times (0,\infty), \tag{1}$$

$$u(.,0) = u_0 \quad \text{in } \mathbb{R}^d, \tag{2}$$

where $f \in C^1(\mathbb{R},\mathbb{R}^d)$ denotes some nonlinear flux function and $u_0 \in L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ some initial data. Here $BV(\mathbb{R}^d)$ is the space of functions with bounded variation (cf. [4]).

In this contribution we focus on the derivation of second order central schemes on a general class of unstructured grids in arbitrary spatial dimensions. The only assumption we use on two subsequent grids is an *overlap assumption*, see below. The second order accuracy is achieved by using a reconstruction and limitation procedure. Furthermore we propose an adaption strategy for the first and second order methods where we use the theoretical a posteriori result of the first order scheme to derive appropriate refinement indicators. We finally implemented the adaptive second order scheme on a particular choice of meshes and demonstrate the performance of the method by numerical experiments.

First and second order central schemes on staggered grids for conservation laws were introduced by Nessyahu and Tadmor in 1990 [8] in one spatial dimension and generalized to particular unstructured grids in two space dimensions by Arminjon and Viallon in 1999 [1]. In [7] first order central schemes were generalized to arbitrary unstructured staggered grids. In addition a priori and a posteriori error estimates were obtained, by interpreting the central scheme on staggered grids as a upwind finite volume scheme in conservation form on the intersection grid accompanied by suitable prolongation and restriction steps. Hence the a priori and a posteriori theory developed in [2] and [6] could be applied.



$\mathcal{T}_h^2 n$ $\quad$ $\mathcal{T}_h^{2n+1}$ $\quad$ $\mathcal{T}_h^{2n,2n+1}$

Figure 1: A possible choice of grids at even and odd timesteps and their intersection.

2 Central schemes on arbitrary unstructured grids

For the discretization of (1),(2) let a sequence of unstructured grids $(\mathcal{T}_h^n)_{n\in\mathbb{N}}$ be given. Here n corresponds to the time level. We assume that each grid $\mathcal{T}_h^n$ is is a non-overlapping partition of $\mathbb{R}^d$ consisting of polyhedrons T_i^n, $i \in I^n \subset \mathbb{N}$, of finite diameter. The sequence of meshes has to satisfy the following regularity *uniformly*: There exists $\alpha > 0$ such that for all $n \in \mathbb{N}$ and all $i \in I^n$

$$\alpha \ \mathrm{diam(T_i^n)}^d \leq |T_i^n|, \quad |\partial T_i^n| \leq \frac{1}{\alpha}\mathrm{diam(T_i^n)}^{d-1}.$$

One possible choice of such grids is illustrated in Fig. 1. In order to go from one grid to another we have to link two consecutive grids $\mathcal{T}_h^n$ and $\mathcal{T}_h^{n+1}$. Thus, for $i \in I^{n+1}$ we define:

$$\begin{aligned} K^{n,n+1}(i) &:= \{j \in I^n | \ T_j^n \cap T_i^{n+1} \neq \emptyset\}, \\ K_\partial^{n,n+1}(i) &:= \{j \in I^n | \ S_{ij}^{n,n+1} := T_j^n \cap \partial T_i^{n+1} \neq \emptyset\}. \end{aligned}$$

The scaled outer normal $\nu_{ij}^{n,n+1}$ to $S_{ij}^{n,n+1}$ is piecewise defined (as the sum of the scaled outer normals on each part of $S_{ij}^{n,n+1}$ contained in a $(d-1)$ dimensional hyperplane) and has the length of $|S_{ij}^{n,n+1}|$.

Two consecutive grids have to satisfy the following overlap assumption which excludes the case that two consecutive grids are equal.

There exists a constant $C_{\mathrm{ov}} > 0$ such that for all $n \in \mathbb{N}$, $i \in I^{n+1}$, $j \in K^{n,n+1}(i)$

$$C_{\mathrm{ov}} \leq \frac{|T_j^n \cap T_i^{n+1}|}{|T_i^{n+1}|} \leq 1, \tag{3}$$

$$\partial T_i^{n+1} \cap \partial T_j^n \quad \text{has dimension at most } d-2. \tag{4}$$

As a theoretical tool we shall need an *intersection grid* $\mathcal{T}_h^{n,n+1} = (T_i^{n,n+1})_{i\in I^{n,n+1}}$ obtained by intersecting each $T_i^n \in \mathcal{T}_h^n$ with each $T_i^{n+1} \in \mathcal{T}_h^{n+1}$. Note that the intersection of two polyhedrons is again a polyhedron. For $i \in I^{n,n+1}$ we denote the set of indices corresponding to neighboring polyhedrons $T_l^{n,n+1}$ by $\mathcal{N}^{n,n+1}(i)$ and the common interface by $S_{il}^{n,n+1}$.

The time axis $[0,\infty)$ is partitioned into intervals $[t^n, t^{n+1}[$, $n \in \mathbb{N}$, of length $k^n := t^{n+1} - t^n$.

2.1 A review of first order central schemes on arbitrary grids

The idea to derive a first order central scheme on the given sequence of unstructured grids for approximating the solution of (1), (2) is the following, see [7]:
Let piecewise constant values $(u_i^n)_{i\in I^n}$, $n \in \mathbb{N}$, be given.

1. Prolongate the values trivially to the intersection grid $\mathcal{T}_h^{n,n+1}$.
2. Perform a upwind finite volume step, e.g. with the Godunov scheme, on the intersection grid.
3. Perform the L^2 projection of these values onto $\mathcal{T}_h^{n+1}$ and get piecewise constant values $(u_i^{n+1})_{i\in I^{n+1}}$.

When considering the overall algorithm the numerical fluxes introduced in the second step reduce to evaluations of the continuous flux f in (1) because of the consistency and conservation of the numerical flux and the overlap assumption on two consecutive grids. The resulting first order Lax-Friedrichs scheme is given explicitly as follows. For given values u_i^n, $i \in I^n$ define values u_i^{n+1}, $i \in I^{n+1}$ by

$$u_i^{n+1} = \sum_{j\in K^{n,n+1}(i)} \frac{|T_j^n \cap T_i^{n+1}|}{|T_i^{n+1}|} u_j^n - \frac{k^n}{|T_i^{n+1}|} \sum_{j\in K_\partial^{n,n+1}(i)} f(u_j^n)\nu_{ij}^{n,n+1}.$$

The advantage of this point of view is that one can analyze central schemes on staggered grids within the same framework as finite volume schemes on a fixed grid with appropriate pre– and post– processing. Particularly one can prove the following a posteriori error estimate (see [7]).

THEOREM 2.1 (A posteriori error estimate)
Assume that the assumptions on the data and the meshes stated above are satisfied and that the CFL-condition

$$k^n V_{[U_m,U_M]} \leq \frac{1}{2}(1-\xi)\alpha^2 h_i^{n,n+1} \quad \xi \in]0,1[, \tag{5}$$

is met, where $V_{[U_m,U_M]}$ is a constant such that $|f'(s)| \leq V_{[U_m,U_M]}$, for all $s \in [U_m, U_M]$. Let $K \subset\subset \mathbb{R}^d \times \mathbb{R}^+$, $\omega = V_{[U_m,U_M]}$ and choose $T, R > 0$ and $x_0 \in \mathbb{R}^d$ such that $T \in]0, R/\omega[$ and $K \subset \bigcup_{0\leq t\leq T} B_{R-\omega t}(x_0) \times \{t\}$. Then we have

$$||u - u_h||_{L^1(K)} \leq T(||u_0 - u_h(.,0)||_{L^1(\{|x-x_0|<R+1\})} + aQ + b\sqrt{Q}), \tag{6}$$

where a and b are computable constants (see [7] for details) and

$$
\begin{aligned}
Q \;:=\;& \frac{1}{2}\sum_{n=0}^{N_0}\sum_{i\in I_D^{n+1}} h_i^{n+1}|T_i^{n+1}| \sum_{j,l\in K^{n,n+1}(i)} \frac{|T_j^n\cap T_i^{n+1}|}{|T_i^{n+1}|}\frac{|T_l^n\cap T_i^{n+1}|}{|T_i^{n+1}|}|u_j^n-u_l^n| \\
&+\sum_{n=0}^{N_0-1} k^n \sum_{i\in I_D^{n+1}} |T_i^{n+1}| \left| u_i^{n+1} - \sum_{j\in K^{n,n+1}(i)} \frac{|T_j^n\cap T_i^{n+1}|}{|T_i^{n+1}|} u_j^n \right| \\
&+6V_{[U_m,U_M]}\sum_{n=0}^{N_0} k^n \sum_{i\in I_D^{n+1}} (h_i^{n+1}+k^n) \sum_{(j,l)\in\mathcal{E}^{n,n+1}(i)} |S_{jl}^{n,n+1}(i)|\,|u_j^n-u_l^n|.
\end{aligned}
$$

2.2 Derivation of the higher order central scheme

In what follows, we will use the same point of view as above to derive second order central schemes on general unstructured staggered grids.

First, we introduce linear reconstruction operators $L^n : PC(\mathcal{T}_h^n) \to PL(\mathcal{T}_h^n)$, where the spaces $PC(\mathcal{T}_h^n)$ and $PL(\mathcal{T}_h^n)$ are defined as

$$
\begin{aligned}
PC(\mathcal{T}_h^n) &:= \{\varphi\in BV(\mathbb{R}^d)|\ \ \varphi|_{T_i} = c_i \in\mathbb{R}, T_i\in\mathcal{T}_h^n\}, \\
PL(\mathcal{T}_h^n) &:= \{\varphi\in BV(\mathbb{R}^d)|\ \ \varphi|_{T_i} \in\mathbb{P}_1, T_i\in\mathcal{T}_h^n\}.
\end{aligned}
$$

Here $\mathbb{P}_1$ denotes the space of linear functions. In addition we assume that L^n fulfills the following properties for all $v_h\in PC(\mathcal{T}_h^n)$, $n\in\mathbb{N}$:

Conservation:

$$\int_{T_i^n} v_h = \int_{T_i^n} L^n v_h, \quad \text{for all } T_i^n\in\mathcal{T}_h^n.$$

Non oscillatory:

$$|L^n v_h|_{BV} \le |v_h|_{BV}, \qquad ||L^n v_h||_{L^\infty} \le ||v_h||_{L^\infty}.$$

Such reconstruction operators are for example the reconstruction of Durlofsky, Engquist, Osher [3] or the reconstruction of Wierse [10] together with some suitable limiters.

Let us derive a continuous version of the second order central scheme which uses generalized Riemann problems, see [4].

Let piecewise constant function u_h^n on $\mathcal{T}_h^n$, $n\in\mathbb{N}$, be given.

1. Form the functions $L^n u_h^n$ which are piecewise linear on $\mathcal{T}_h^n$.

2. Prolongate the functions trivially to the intersection grid $\mathcal{T}_h^{n,n+1}$.

3. Let v denote the solution of the generalized Riemann problem

$$\partial_t v + \mathrm{div}\, \mathrm{f(v)} = 0 \quad \text{in } \mathbb{R}^{\mathrm{d}} \times (\mathrm{t^n}, \mathrm{t^{n+1}}), \quad \mathrm{v}(., \mathrm{t^n}) = \mathrm{u_h^n} \quad \text{in } \mathbb{R}^{\mathrm{d}}.$$

Integration of this PDE gives for $i \in I^{n,n+1}$

$$\begin{aligned} v_i^{n+1} \quad &:= \quad \frac{1}{|T_i^{n,n+1}|} \int_{T_i^{n,n+1}} v(\cdot, t^{n+1}) = \frac{1}{|T_i^{n,n+1}|} \int_{T_i^{n,n+1}} L^n u_h^n \\ & \quad - \frac{k^n}{|T_i^{n,n+1}|} \sum_{l \in \mathcal{N}^{n,n+1}(i)} \int_{t^n}^{t^{n+1}} \int_{S_{il}^{n,n+1}} f(v(\sigma, t)) n_{il} d\sigma dt, \end{aligned}$$

where n_{il} denotes the unit outer normal to $S_{il}^{n,n+1}$.

4. Perform the L^2 projection of these values onto $\mathcal{T}_h^{n+1}$ and get piecewise constant values $(u_i^{n+1})_{i \in I^{n+1}}$.

By the same reasoning as in the first order case we get for the overall scheme the following formula:

$$\begin{aligned} u_i^{n+1} \quad &= \quad \frac{1}{|T_i^{n+1}|} \sum_{j \in K^{n,n+1}(i)} \int_{T_j^n \cap T_i^{n+1}} L^n u_h^n \\ & \quad - \frac{1}{|T_i^{n+1}|} \sum_{j \in K_\partial^{n,n+1}(i)} \int_{t^n}^{t^{n+1}} \int_{S_{ij}^{n,n+1}} f(v(\sigma, t)) n_{ij} d\sigma dt. \end{aligned}$$

In order to get a feasible numerical method we have to approximate the flux integrals by appropriate quadrature rules. Similar to Nessyahu and Tadmor [8] we assume the time step to be small enough such that the solution of the generalized Riemann problem is smooth across the edges $S_{ij}^{n,n+1}$. Hence, the solution satisfies the PDE in a classical pointwise sense on those interfaces. Therefore, we can use the midpoint rule to approximate the time integral

$$\begin{aligned} & \int_{t^n}^{t^{n+1}} \int_{S_{ij}^{n,n+1}} f(v(\sigma, t)) n_{ij} d\sigma dt \\ & = \quad k^n \int_{S_{ij}^{n,n+1}} f(v(\sigma, t^n + 0.5k^n)) n_{ij} d\sigma + O((k^n)^3). \end{aligned}$$

In order to express $v(\sigma, t^n + 0.5k^n)$ in terms of values at time level t^n we use Taylor's expansion and get for $x \in S_{ij}^{n,n+1}$

$$\begin{aligned} v(x, t^n + 0.5k^n) &= v(x, t^n) + \frac{1}{2}k^n \partial_t v(x, t^n) + O((k^n)^2) \\ &= v(x, t^n) - \frac{1}{2}k^n \mathrm{div} f(v(x, t^n)) + O((k^n)^2) \\ &= v(x, t^n) - \frac{1}{2}k^n \sum_{i=1}^{d} f_i'(L^n u_h^n(x))\partial_{x_i} L^n u_h^n(x) + O((k^n)^2). \end{aligned}$$

Using again an appropriate quadrature rule (e.g. trapezoidal rule) to approximate the spatial integral we end up with the following predictor-corrector form of our second order central scheme which is the generalization of the Nessyahu-Tadmor scheme to unstructured staggered grids in any spatial dimension.

Definition (Second order staggered Lax-Friedrichs scheme) Define an approximation u_h to the solution of (1), (2) by the following scheme.

For $i \in I^0$ set

$$u_i^0 := \frac{1}{|T_i^0|}\int_{T_i^0} u_0(x)\mathrm{d}x.$$

For given values u_i^n, $i \in I^n$ define values u_i^{n+1}, $i \in I^{n+1}$ by

$$\begin{aligned} u_{ijk}^{n+\frac{1}{2}} &= u_j^n - \frac{1}{2}k^n \sum_{i=1}^{d} f_i'(L^n u_h^n(z_{ijk}))\partial_{x_i} L^n u_h^n(z_{ijk}), \\ u_i^{n+1} &= \frac{1}{|T_i^{n+1}|}\sum_{j\in K^{n,n+1}(i)}\int_{T_j^n\cap T_i^{n+1}} L_j^n u \\ &\quad - \frac{k^n}{|T_i^{n+1}|}\sum_{j\in K_\partial^{n,n+1}(i)} |S_{ij}^{n,n+1}| \sum_{k=1}^{L(i,j)} \omega_{ijk} f(u_{ijk}^{n+\frac{1}{2}})\nu_{ij}^{n,n+1}, \end{aligned}$$

where for all $i \in I^{n+1}$ and $j \in K_\partial^{n,n+1}(i)$ $L(i,j)$ denotes the number of integration points z_{ijk} on $S_{ij}^{n,n+1}$ with positive weights ω_{ijk} such that $\sum_{k=1}^{L(i,j)} \omega_{ijk} = 1$. Finally, also in the second order case we define the discrete solution as

$$u_h(x,t) = u_i^n \quad \text{for } t \in [t^n, t^{n+1}[\text{ and } x \in T_i^n.$$

3 Implementation and numerical experiments

We finally implemented the second order scheme in two spatial dimensions based on a primal triangular mesh for even time steps and on a dual - Donald type - mesh for odd time steps (see Fig. 1). The grid adaption is performed solely at the even timesteps on the primal triangular grid where we use the theoretical a posteriori result of the first order method for our adaption strategy (see also [6]). The corresponding adaptive dual mesh is then implicitly given as the dual mesh of the adaptive triangulation. This particular choice of meshes allows us to use the well known MUSCL–type reconstruction of Durlofsky, Engquist and Osher [3] together with some modified superbee limiter on the primal triangulation and a reconstruction of Sonar [9] with the Barth-Jesperson limiter on the dual mesh in order to obtain the second order method.

The following numerical experiments show that the second order central scheme produces at least as good results as an second order upwind method does (see also [5]). In addition, a comparison with calculations on a uniform grid demonstrate the efficiency of the adaptive scheme.

3.1 Linear advection problems

Let us consider the linear conservation law (1) where $f(u) = (u, 0)^\top$ together with either discontinuous or smooth initial data. For the discontinuous case we choose the characteristic function of a square given by

$$u_0(x) = \begin{cases} 1, & \text{if} \quad 0.3 \le x_1 \le 0.4, 0.2 \le x_2 \le 0.3, \\ 0, & \text{else} \end{cases} ,$$

while in the smooth case we choose a global C^1–function given by

$$u_0(x) = \begin{cases} 0.1(2r^3 - 3r^2 + 1), & \text{if} \quad r := 10\sqrt{(x_1 - 0.3)^2 + (x_2 - 0.25)^2} \le 1 \\ 0, & \text{else} \end{cases} .$$

In both cases the exact solution is $u(x,t) = u_0(x_1 - t, x_2)$.

Figures 2 and 4 show the numerical solutions of the adaptive higher order central scheme in both cases, when the initial condition is discontinuous or smooth. In the corresponding diagrams (Fig. 3 and Fig. 5) we compare the uniform and adaptive higher order central scheme with an higher order upwind scheme. The results demonstrate, that the central scheme on uniform meshes behaves at least as good

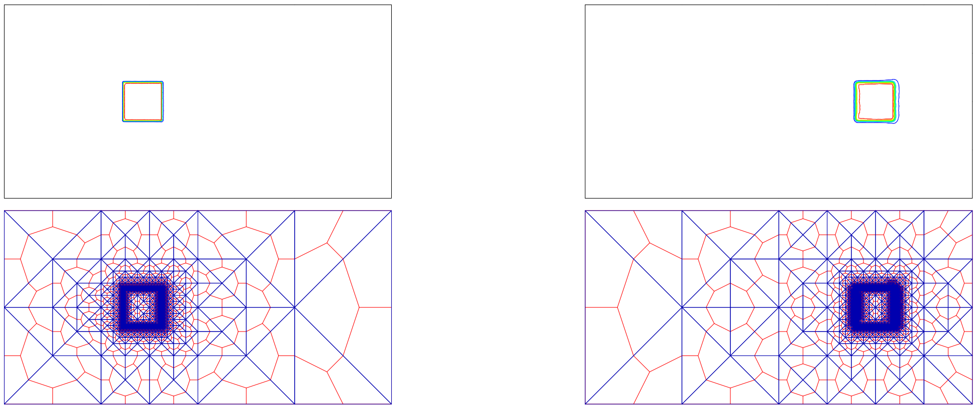


Figure 2: Linear advection of the characteristic function of a square ($t = 0.0$ on the left hand side and $t = 0.4$ on the right hand side). The upper pictures show the isolines of the solution, while the staggered adaptive computational grids are shown at the bottom.

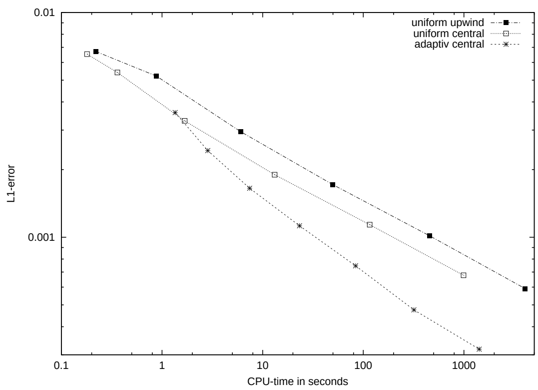


Figure 3: L^1 error versus CPU time for the linear advection of a characteristic function. Comparison between uniform second order upwind method, uniform second order central scheme and adaptive second order central scheme.

as an second order upwind method does. In addition, one can gain a lot in efficiency by using adaptive methods.

3.2 A Burgers type problem

Let us consider the nonlinear conservation law (1) together with the initial condition (2) in $\mathbb{R}^2$, where

$$f(u) = \begin{pmatrix} u^2 \\ u^2 \end{pmatrix} \qquad \text{and} \qquad u_0(x) = \begin{cases} 2, & \text{if} \quad \frac{x_1+x_2}{2} - 0.5 \leq 0 \\ 1, & \text{if} \quad \frac{x_1+x_2}{2} - 0.5 > 0 \end{cases} .$$

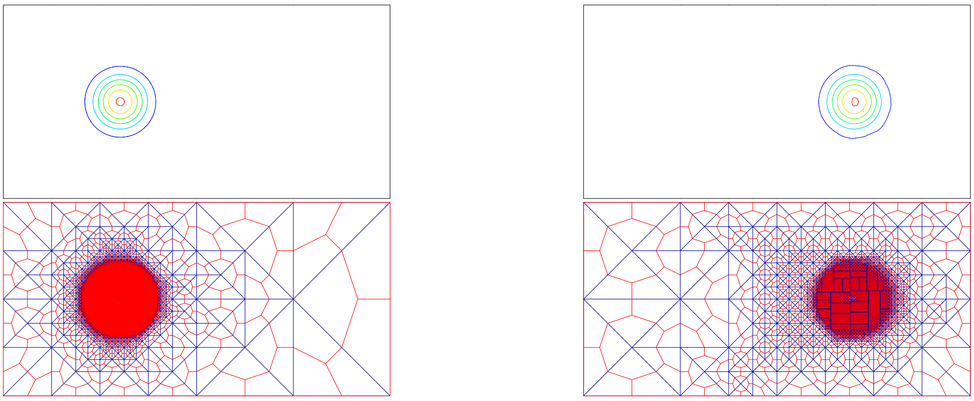


Figure 4: Linear advection of a smooth (C^1) function ($t = 0.0$ on the left hand side and $t = 0.4$ on the right hand side). The upper pictures show the isolines of the solution, while the staggered adaptive computational grids are shown at the bottom.

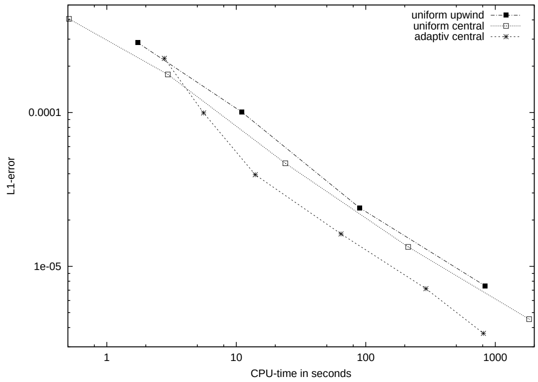


Figure 5: L^1 error versus CPU time for the linear advection of a smooth function. Comparison between uniform second order upwind method, uniform second order central scheme and adaptive second order central scheme.

Then the exact solution of this Burgers type problem is

$$u(x,t) = \begin{cases} 2, & \text{if} \quad \frac{x_1+x_2}{2} - 0.5 \leq 3t \\ 1, & \text{if} \quad \frac{x_1+x_2}{2} - 0.5 > 3t \end{cases} .$$

In Figure 6 the adaptive refined dual grid of the second order central scheme together with a color shading of the discrete solution is shown at $t = 0$ and $t = 0.1$ At the left hand side of Figure 6 we compare the uniform and adaptive second order central scheme with an second order upwind scheme. Also in this case the efficiency of the adaptive central scheme is shown.

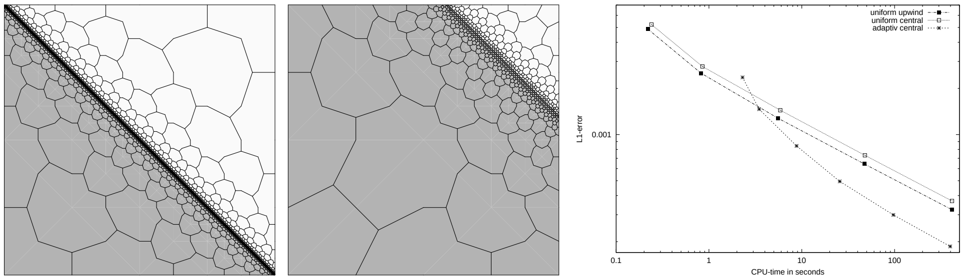


Figure 6: Burgers type convection of a shock ($t = 0.0$ on the left hand side and $t = 0.1$ in the middle). The pictures show a color shading of the solution (red = 2.0, blue = 1.0) together with the adaptive dual computational grid. At the right hand side the L^1 error versus CPU time for the Burgers type convection of a step function is shown. Comparison between uniform second order upwind method, uniform second order central scheme and adaptive second order central scheme.

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